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IDENTITIES OF F-ALGEBRAS

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To my family

Alvin, Henry, Arthur and Edith

ABSTRACT

The basic theory of varieties and T-ideals is developed. Processes for obtaining identities for a ring are investigated with partial success. Some examples are given. If f is an identity and F is of characteristic zero or greater then the degree of f in each variable, then f is based on multilinear identities. If f is of characteristic zero and T_1 and T_2 are finitely generated, then $T_1(T_2)$ is finitely generated.

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CHAPTER I

T-IDEALS AND VARIETIES

In this thesis I shall endeavor to show some of the problems in calculating polynomial identities of F-algebras. I shall develop the basic theory of varieties and T-ideals.

Let F be a commutative ring with unity element 1. Let X be a countably infinite set of noncommuting indeterminates. Denote by $F[X]$ the free associative F-algebra generated by the set X. $F[X]$ is the ring of all polynomials in noncommuting indeterminates of X with coefficients in F and 0 constant term.

N. Jacobson [9] defines $F[X]$ more generally by not requiring $F[X]$ to be associative. Then all algebras satisfying $x_1(x_2x_3) - (x_1x_2)x_3$ are associative, those satisfying $x_1x_2 + x_2x_1$ are Jacobson algebras, and those satisfying $[[x,y],z] + [[y,z],x] + [[z,x],y]$, where $[x,y] = xy - yx$, are Lie algebras.

The ideal $T \subseteq F[X]$ is a <u>T-ideal</u> if for any F-endomorphism σ of $F[X]$, $T^{\sigma} \subseteq T$.

Lemma 1: $\bigcap_{\lambda} T_{\lambda}$ is a T-ideal for λ belonging to any index set.

Proof: $T = \bigcap_{\lambda} T_{\lambda}$ is an ideal. Let σ be an F-endomorphism of $F[X]$. For all λ, $T \subseteq T_{\lambda}$ implies $T^{\sigma} \subseteq T_{\lambda}$ therefore $T^{\sigma} \subseteq \bigcap_{\lambda} T_{\lambda}$ and thus T is a T-ideal.

Given a set $A \subset F[X]$ we say $T\langle A\rangle$ is the T-ideal <u>generated</u> by A if $T\langle A\rangle$ is the smallest T-ideal containing A. Also A is said to <u>generate</u> $T\langle A\rangle$. We also say A is a <u>basis</u> for $T\langle A\rangle$ and

that $f \in T\langle A\rangle$ is <u>based</u> on A.

$T\langle A\rangle$ is well defined since $T\langle A\rangle = \bigcap_{A \subset T} T$, T a T-ideal in a fixed $F[X]$; and, by Lemma 1, $T\langle A\rangle$ is a T-ideal.

Lemma 2: T-ideals have the following properties:

i. $T\langle T\langle A\rangle\rangle = T\langle A\rangle$.

ii. $A \subseteq B$ implies $T\langle A\rangle \subseteq T\langle B\rangle$.

iii. $C \subseteq T\langle A\rangle$ implies $T\langle C\rangle \subseteq T\langle A\rangle$.

iv. $C \subseteq T\langle A\rangle$ and $A \subseteq C$ implies $T\langle A\rangle = T\langle C\rangle$.

The proofs of these are obvious; i and ii come from the definition, iii follows from i and ii, and iv from ii and iii.

Since $T\langle A\rangle$ is the smallest T-ideal containing A, the ideal $T\langle A\rangle$ is the two-sided ideal generated by

$$\{a^\sigma : a \in A, \sigma \text{ is an F-endomorphism of } F[X]\} .$$

Any element t of $T\langle A\rangle$ has the form

$$t = \sum C_i \, \alpha_i^{\epsilon_{i,1}} \, a_i(\gamma_{i,1},\ldots,\gamma_{i,n(i)})\beta_i^{\epsilon_{i,2}}$$

where

$$\alpha_i, \beta_i, \gamma_{i,j} \in F[X], \; a_i = a_i(x_{i,1},\ldots,x_{i,n(i)}) \in A$$

$$\epsilon_{i,j} = 0 \text{ or } 1, \quad C_i \in F .$$

Every associative ring can be considered as an algebra over the integers. Unless otherwise specified any ring shall be an associative ring. A ring R, considered as an F-algebra for some F

satisfies a polynomial identity $f \in F[X]$ if $f \not\equiv 0$ and for any homomorphism $\phi : F[X] \to R$, $\phi(f) = 0$. In other words, if $f = f(x_1,\ldots,x_n)$ and $r_1,\ldots,r_n \in R$ then $f(r_1,\ldots,r_n) = 0$. We also say $f(R) = 0$. If f is not an identity for R, then $f(R) = \{f(r_1,\ldots,r_n) : r_1,\ldots,r_n \in R\}$.

For example, commutative rings satisfy $xy - yx$ and nilpotent rings of exponent t satisfy $x_1x_2\ldots x_t$.

Lemma 3: Let $A \subset F[X]$ be any set of identities. The ring $F[X]/T\langle A\rangle$ satisfies exactly the identities of $T\langle A\rangle$.

Proof: Let $p = p(x_1,\ldots,x_n) \in T\langle A\rangle$. Let ϕ be an arbitrary homomorphism from $F[X]$ to $F[X]/T\langle A\rangle$. We have $\phi(p) = p(\phi(x_1),\phi(x_2),\ldots,\phi(x_n)) \in T\langle A\rangle$. Since $\phi(x_i)$ are arbitrary elements of $F[X]/T\langle A\rangle$, p is an identity for $F[X]/T\langle A\rangle$.

On the other hand, suppose $p \notin T\langle A\rangle$. Let ϕ be the natural homomorphism of $F[X]$ onto $F[X]/T\langle A\rangle$. The kernel of ϕ is $T\langle A\rangle$, which implies $0 \neq \phi(p) = p(\phi(x_i))$. Thus p is not an identity for $F[X]/T\langle A\rangle$.

Denote by $\underline{I(R)}$ the set of all polynomial identities of a ring R; and by $I(\mathcal{A})$ the set of all polynomial identities satisfied by each member of a class $\mathcal{A}$ of F-algebras.

Lemma 4: Let R be an F-algebra; then $I(R)$ is a T-ideal of $F[X]$.

Proof: Let $f \in T\langle I(R)\rangle$. Then

$$f = \sum c_i \alpha_i^{\epsilon_{i,1}} p_i(\gamma_{i,j}) \beta_i^{\epsilon_{i,2}}$$

where

$p_i \in I(R)$, $c_i \in F$, $\alpha_i, \beta_i, \gamma_{i.k} \in F[X]$ and $\epsilon_{i,j} = 0$ or 1.

But for any map $\phi : F[X] \to R$

$$\phi(\gamma_{i,j}) \in R, \quad \text{so let} \quad \phi(\gamma_{i,j}) = r_{i,j} \, .$$

Then $\phi(p_i(\gamma_{i,j})) = p_i(r_{i,j}) = 0$. Thus

$$\phi(f) = \sum c_i \, \phi(\alpha_i^{\epsilon_{i,1}}) \cdot 0 \cdot \phi(\beta_i^{\epsilon_{i,2}}) = 0 \, ;$$

so f is an identity of R and

$$T<I(R)> = I(R) \, .$$

A <u>variety</u> $\mathcal{V}$s of F-algebras is the class of all F-algebras satisfying a given set S of polynomial identities.

Lemma 5: The set of all polynomial identities satisfied by every member of a variety of F-algebras is a T-ideal in $F[X]$.

Proof: Let A be the set of all identities of the variety $\mathcal{V}$. Let $R \in \mathcal{V}$. Then Lemmas 2 and 4 assure us that since $A \subset I(R)$, $T<A> \subset I(R)$. Thus $T<A> \subset I(R)$ for all $R \in \mathcal{V}$, so $T<A> \subset A$ and A is a T-ideal.

Lemma 6: There exists a one-to-one correspondence between varieties of F-algebras and T-ideals of $F[X]$.

Proof: By Lemma 3, distinct T-ideals yield distinct varieties. By Lemma 5, distinct varieties yield distinct T-ideals.

Lemma 7: Let $R = \prod_{\lambda \in \Lambda} A_\lambda$ where A_λ are F-algebras and Λ is an index set. Then $I(R) = \bigcap_{\lambda \in \Lambda} I(A_\lambda)$.

Proof: Let $\alpha \in \Lambda$ and consider the injection map $\sigma_\alpha : A_\alpha \to R$ where $\sigma_\alpha(a_\alpha)$ is a_α in the α^{th} component and 0 elsewhere. Thus the kernel of σ_α is $(0) \subset A_\alpha$. Let $p \in I(R)$ and $a_{\alpha,1},\ldots,a_{\alpha,n} \in A_\alpha$. Then $\sigma_\alpha(p(a_{\alpha,1},\ldots,a_{\alpha,n})) = p(\sigma_\alpha(a_{\alpha,1}),\ldots,\sigma_\alpha(a_{\alpha,n})) = 0$ since p was in $I(R)$. Since the kernel of σ_α is (0), it must be that $p(a_{\alpha,1},\ldots,a_{\alpha,n}) = 0$ and p is an identity for A. Thus $I(R) \subseteq \bigcap_{\lambda\in\Lambda} I(A_\lambda)$.

Now let $p \in \bigcap_{\lambda\in\Lambda} I(A_\lambda)$ and $r_i \in R$. Then the λ^{th} component of $p(r_i)$ is $p(r_{\lambda,i})$. But for all λ, $p \in I(A_\lambda)$ and thus $p(r_{\lambda,i}) = 0$. Therefore $p(r_i) = 0$ and $p \in I(R)$. Thus $\bigcap_{\lambda\in\Lambda} I(A_\lambda) = I(R)$.

Unfortunately this lemma does little to help us find the identities of a specific ring. Knowing the basis of two T-ideals, T_1 and T_2, does not seem to give us a basis for $T_1 \cap T_2$. Consider the following:

$$T_1 = T\langle xyz\rangle$$

$$T_2 = T\langle xy-yx\rangle.$$

Both ideals are generated by one identity. Let $T_3 = T\langle (xy-yx)z, z(xy-yx)\rangle$. By inspection $T_3 \subset T_1 \cap T_2$. These two identities are independent for the ring of matrices

$$R = \left\{ \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f \end{pmatrix} : a, b, c, d, f \in \text{rationals} \right\}$$

satisfies $z(xy-yx)$ but not $(xy-yx)z$. Similarly the ring $R' = \{\text{transpose of } r : r \in R\}$ satisfies the second identity but not the first. Thus two blended identities are needed in a generator of $T_1 \cap T_2$. I cannot show that $T_1 \cap T_2 \subset T_3$.

Let A be a class of F-algebras. Define $\underline{SA} = \{R' : R' \leq R \in A\}$ i.e. the class of subalgebras of algebras in A. The class A is S-closed if $SA = A$. Similarly $\underline{QA} = \{R' : R'$ is a homomorphic image of $R \in A\}$. The class A is Q-closed if $QA = A$. Define $\Pi A = \{\prod_\alpha R_\alpha : \{R_\alpha\} \subset A\}$, all direct products of members of A. The class A is Π-closed if $\Pi A = A$. The class A is closed if it is S-, Q- and Π-closed.

The closure of a class A of F-algebras, denoted by $c\ell(A)$, is the class of all F-algebras obtained from members of A by taking a finite number of the operations of taking subalgebras, homomorphic images and direct products. Obviously $c\ell(A)$ is closed, and any closed class containing A also contains $c\ell(A)$.

Lemma 8: Let A be a class of F-algebras. Then the identities of QA, SA, ΠA and $c\ell(A)$ are exactly the identities of A.

Proof: Let R' be a subalgebra of R, a member of A. All elements of R' are also elements of R so must satisfy all identities of R. Thus $I(R') \supset I(R) \supset I(A)$, and $I(SA) \supset I(A)$.

Let R' be a homomorphic image of $R \in A$. Then there is a homomorphism, ϕ, from R onto R' such that $\phi(R) = R'$. Let f be an identity of R. Then $f(R) = (0)$, so $(0) = \phi(f(R)) = f(\phi(R))$ and f is an identity for R'. Thus $I(R') \supset I(R) \supset I(A)$.

Let $\{R_\lambda\} \subset A$ and $R = \prod_\lambda R_\lambda$. By Lemma 7 $I(R) = \bigcap_\lambda I(R\ ) \supset I(A)$.

Since A is contained in SA, QA and ΠA, the identities of each class must be equal.

Since $c\ell(A)$ is formed by taking a finite number of operations and $I(SA) = I(QA) = I(\Pi A) = I(A)$, then by induction $I(c\ell(A)) = I(A)$.

Corollary 8: A variety, A, is a closed class.

Proof: Since QA, SA and ΠA satisfy the identities of A they must be contained in A. Thus A is closed.

The converse of Corollary 8 is more difficult to prove.

Lemma 9: Given a closed class, M, of F-algebras then there exists $R \in M$ such that $I(R) = I(M)$.

Proof: Index the elements of $F[X]\backslash I(M)$ by Λ. For each $\lambda \in \Lambda$ and $f_\lambda \in F[X]\backslash I(M)$ there is $R_\lambda \in M$ such that f_λ is not an identity for R_λ. Let $R = \prod_{\lambda\in\Lambda} R_\lambda$. Since M is Π-closed, $R \in M$.

The ideal $I(R) = \bigcap_{\lambda\in\Lambda} I(R_\lambda)$ by Lemma 7. By our construction, for any $f_\lambda \in F[X]\backslash I(M)$, $f_\lambda \notin I(R_\lambda)$. Thus $f_\lambda \notin \bigcap_{\lambda\in\Lambda} I(R_\lambda) = I(R)$. Therefore $I(M) \supset I(R)$. But since $R \in M$, $I(R) \supset I(M)$. Thus they are equal.

Lemma 10: Given a class, A, of F-algebras then $c\ell(A) = QS\Pi A$.

Proof: By definition of $c\ell(A)$, $QS\Pi A \subset c\ell(A)$. Note that each operation is idempotent: $QQA = QA$, $\Pi\Pi A = \Pi A$ and $SSA = A$. Thus we need the show that $QS\Pi A$ is Q-, S-, Π-closed.

By above, $QQS\Pi A = QS\Pi A$ and $QS\Pi A$ is Q-closed.

Let $R \in QS\Pi A$, and R' be a subalgebra of R. Then there exists an $R'' \in S\Pi A$ and a homomorphism, σ, of R'' onto R such that $\sigma(R'') = R$. But since σ is a homomorphism, the preimage of any subalgebra of R is a subalgebra of R''; thus there is $M < R''$ such that $\sigma(M) = R'$. The fact that $M < R'' \in S\Pi A$ and S an idempotent operation implies $M \in S\Pi A$ and $\sigma(M) = R' \in QS\Pi A$. Thus $QS\Pi A$ is S-closed.

Now let $\{R_\lambda\} \subset QS\Pi A$ and $\prod_\lambda R_\lambda = R$. For each λ, $R_\lambda \in QS\Pi A$ implies there is $R'_\lambda \in S\Pi A$ and a homomorphism σ_λ from R'_λ onto R_λ. Further there is $R''_\lambda \in \Pi A$ such that $R'_\lambda \leq R''_\lambda$. $\prod_\lambda R''_\lambda \in \Pi A$ since each $R'' \in \Pi A$. Also $R'_\lambda \leq \Pi R''_\lambda$ since for each λ, $R'_\lambda \leq R''_\lambda$. Thus $\prod_\lambda R'_\lambda \in S\Pi A$. Define $\sigma : \Pi R_\lambda \to \Pi R_\lambda$ such that for each λ, $\sigma(R'_\lambda) = \sigma_\lambda(R'_\lambda)$. Thus $\prod_\lambda R_\lambda = \prod_\lambda \sigma_\lambda(R'_\lambda) = \sigma(\prod_\lambda R'_\lambda) \in QS\Pi A$. Therefore $QS\Pi A$ is Π-closed and $c\ell(A) = QS\Pi A$.

Let Y be a set of noncommuting indeterminates and $F[Y]$ the free associative F-algebra generated by Y. Let I be a set in $F[X]$. Then $\underline{T_Y}\langle I\rangle = \{\tau(p) : p \in T\langle I\rangle$ and $\tau \in \mathrm{Hom}(F[X],F[Y])\}$. Let $r \in F[Y]$ and $m \in \sum_{\tau\in\mathrm{Hom}(F[X],F[Y])} \tau\, T\langle I\rangle$. Then $m = \sum_{i\in I} \tau_i(t_i)$ where I is finite, $\tau_i \in \mathrm{Hom}(F[X],F[Y])$ and $t_i \in T\langle I\rangle$; so $\{\tau_i(t_i)\} \in T_Y\langle I\rangle$. Each $t_i = t_i(x_{i,1},\ldots,x_{i,n(i)})$. Let $\tau_i(x_{i,j}) = g_{i,j} \in T[Y]$, where $j = 1,\ldots,n(i)$ and $g_{i,j} = g_{i,j}(y_{i,j,1},\ldots,y_{i,j,m(i,j)})$. The set $\{(i,j,k)\} = A$ is finite. Consider A as an index set for $X_o \subset X$, where the cardinality of A and C_o are the same. Let $\sigma \in \mathrm{Hom}(F[X], F[Y])$ such that $\sigma : x_a \to y_{i,j,k}$ for $x_a \in X_o$, and $\sigma : x_o \to r$ for

$x_o \in X \backslash X_o$. Since $t_i \in T\langle I\rangle$, $\sum_i t_i(g_{i,j}(x_{i,j,1},\ldots,x_{i,j,m(i,j)}) =$ $t_o \in T\langle I\rangle$ and $x_o t_o \in T\langle I\rangle$. So $m = \sigma(t_o) \in T_Y\langle I\rangle$, $rm = \sigma(x_o t_o) \in T_Y\langle I\rangle$ and $-m = \sigma(-t_o) \in T_Y\langle I\rangle$. So $T_Y\langle I\rangle$ is an ideal. Further if $\tau(p) \in T_Y\langle I\rangle$ and σ is an endomorphism of $F[Y]$, then since $\sigma\tau \in \mathrm{Hom}(F[X],F[Y])$, $\sigma\tau(p) \in T_Y\langle I\rangle$. So $T_Y\langle I\rangle$ is a T-ideal and we shall say that I generates $T_Y\langle I\rangle$.

The <u>I-free F-algebras</u> are $F[Y]/T_Y\langle I\rangle$ where Y is an arbitrary set.

If Y is countable then one can let τ be fixed as the canonical homomorphism taking $x_i \to y_i$ until Y is exhausted, and the rest of X to 0. That is, $T_Y\langle I\rangle$ is simply $\tau(T\langle I\rangle)$. This is possible, since to any endomorphism σ of $F[Y]$ there corresponds an endomorphism σ' of $F[X]$ such that $\sigma\tau' = \sigma\tau$, and the diagram commutes. Then for $p \in T\langle I\rangle$, $\sigma'(p) \in T\langle I\rangle$; so $\sigma\tau(p) = \tau\sigma'(p) \in T_Y\langle I\rangle$, and $T_Y\langle I\rangle$ is a T-ideal.

$$\begin{tikzcd} F[X] \arrow[r, "\tau"] \arrow[d, "\sigma'"'] & F[Y] \arrow[d, "\sigma"] \\ F[X] \arrow[r, "\tau"] & F[Y] \end{tikzcd}$$

Theorem 11: Let R be an F-algebra satisfying a polynomial identity. Let I be the identities of R. Then $c\ell(R)$ contains all I-free F-algebras.

Proof: Let Y be an arbitrary set of noncommuting indeterminantes. Then $F[Y]/T_Y\langle I\rangle$ is an I-free F-algebra.

Let $R_Y = \prod_{\sigma \in \mathrm{Hom}(F[Y],R)} R_\sigma$ where $R_\sigma = R$. Then $R_Y \in c\ell(R)$.

Define $\tau : F[Y]/T_Y\langle I\rangle \to R_Y$ by

$$\tau(f(y)+T_Y\langle I\rangle) = \prod_\sigma \sigma(f(y)), \quad \text{where} \quad f(y) \in F[Y].$$

If $f(y) \in T_Y\langle I\rangle$, then there exists $\gamma \in \mathrm{hom}(F[X],F[Y])$ and $g \in T\langle I\rangle$ such that $\gamma(g(x)) = f(y)$. Hence $\sigma(f(y)) = \sigma\gamma(g(x)) = 0$, since $\sigma\gamma$

is a homomorphism from $F[X]$ to R. Thus $(f(y)+T_Y\langle I\rangle) = 0$.

If $f(y) \notin T_Y\langle I\rangle$, then there exists $\sigma \in \mathrm{Hom}(F[Y],R)$ such that $\sigma(f(y)) \neq 0$. Thus $\tau(f(y)+T_Y\langle I\rangle) \neq 0$, and kernel of τ is (0).

Thus τ is one-to-one and $F[Y]/T_Y\langle I\rangle$ is isomorphic to a subalgebra of T_Y. So it, too, is a member of $c\ell(R)$ and $c\ell(R)$ contains all I-free F-algebras.

Lemma 12: Let I be a set of identities. Let $\mathcal{A}$ be the class of all I-free F-algebras. Then $c\ell(\mathcal{A}) \supset \mathcal{V}_I$, the variety of F-algebras satisfying the identities of I.

Proof: Let $A \in \mathcal{V}_I$. For some set Y of noncommuting indeterminates and the free F-algebra, $F[Y]$, generated by Y, there exists a homomorphism $\psi : F[Y] \to A$ such that ψ is onto A and the kernel of ψ is N. Form the T-ideal $T_Y\langle I\rangle$ in $F[Y]$ generated by I. Then $F[Y]/T_Y\langle I\rangle$ is an I-free F-algebra and thus in $\mathcal{A}$.

Let μ be the canonical homomorphism from $F[Y]$ onto $F[Y]/T_Y\langle I\rangle$. Let $p \in T_Y\langle I\rangle$. Then there exists $p' \in T\langle I\rangle$ and $\tau \in \mathrm{Hom}(F[X],F[Y])$ such that $p = \tau(p')$ and $\mu(p) = \mu\tau(p')$. But $\mu\tau \in \mathrm{Hom}(F[X],A)$, so $\mu\tau(p') = 0 = \mu(p)$ and $T_Y\langle I\rangle \subset N$.

Thus there exists $\phi : F[Y]/T_Y\langle I\rangle \to A$ so that for $f(y) \in F[Y]$, $\phi(f(y)+T_Y\langle I\rangle) = \psi f(y)$ and $\phi(\mu(f(y)) = \psi(f(y))$.

Therefore A is a homomorphic image of an I-free F-algebra and $A \in c\ell(\mathcal{A})$ and $\mathcal{V}_I \subset c\ell(\mathcal{A})$.

Theorem 13: The closed classes of F-algebras are exactly the varieties of F-algebras: i.e., $\mathcal{M}$ is a variety if and only if $\mathcal{M}$

is a closed class of F-algebras.

Proof: A variety is closed by Corollary 8.

Let M be a closed class of F-algebras, then $c\ell(M) = M$. Lemma 9 says there exists $R \in M$ such that $I(R) = I(M)$.

Let A be the class of all $I(M)$-free F-algebras. Then by Theorem 11, $A \subset c\ell(R) \subset c\ell(M) = M$.

Let V be the variety of F-algebras satisfying $I(M)$. By Lemma 12 $V \subset c\ell(A) \subset c\ell(M) = M$. Certainly any algebra in M satisfies $I(M)$; so $M \subset V$ and $M = V$.

Lemma 14: Let M_λ be varieties of F-algebras for F fixed and λ element of some indexing set. Then $\bigcap_\lambda M_\lambda$ is a variety.

Proof: Let $N = \bigcap_\lambda M_\lambda$. Then for all λ, $c\ell(N) \subset M_\lambda$ since M_λ is a variety and thus by Theorem 13 is a closed class.

Thus $c\ell(N) \subset \bigcap_\lambda M_\lambda = N$ and N is a closed class of F-algebras and by Theorem 13 is a variety.

One can construct the lattice of varieties of F-algebras by using intersection and the closure of union. Similarly, one can construct the lattice of T-ideals in $F[X]$ by using intersections and the T-ideal generated by the union of T-ideals.

Theorem 15: There is an isomorphism between the dual of the lattice of varieties of F-algebras and the lattice of T-ideals of $F[X]$.

Proof: Let M_1 and M_2 be two varieties of F-algebras, and T_1 and T_2 be the T-ideals of their identities respectively. Let T_3 be the identities of the class $M_1 \cup M_2$. Certainly

$T_1 \cap T_2 \subset T_3$. By Lemma 9, there exist $R_i \in \mathcal{M}_i$ such that $I(R_i) = T_i$, for $i = 1$ or 2. Thus $T_3 \subset I(R_1) \cap I(R_2) = T_1 \cap T_2$, and $T_3 = T_1 \cap T_2$ which by Lemma 8 is equal to the identities of $c\ell(\mathcal{M}_1 \cup \mathcal{M}_2)$.

Let T_4 be the identities of $\mathcal{M}_1 \cap \mathcal{M}_2$. For all $R \in \mathcal{M}_1 \cap \mathcal{M}_2$, $I(R) \supset T_1 \cup T_2$. But $I(R)$ is a T-ideal so $I(R) \supset T\langle T_1 \cup T_2\rangle$. Therefore $T_4 \supset T\langle T_1 \cup T_2\rangle$.

Now $T\langle T_1 \cup T_2\rangle \supset T_i$ for $i = 1$ or 2. Thus $R = F[X]/T\langle T_1 \cup T_2\rangle \in Q\mathcal{M}_i \subset \mathcal{M}_i$, and so $R \in \mathcal{M}_1 \cap \mathcal{M}_2$. But $I(R) = T\langle T_1 \cup T_2\rangle$ so $T_4 \subset T\langle T_1 \cup T_2\rangle$ and $T_4 = T\langle T_1 \cup T_2\rangle$. Therefore there exists an isomorphism between the dual of the lattice of varieties of F-algebras and the lattice of T-ideals of $F[X]$.

Let $\underline{\underline{T_1+T_2}} = \{t_1+t_2 : t_1 \in T_1$ and $t_2 \in T_2\}$ where T_1 and T_2 are T-ideals of $F[X]$.

Lemma 16: $T\langle T_1 \cup T_2\rangle = T_1+T_2$. Further if $T_1 = T\langle A\rangle$ and $T_2 = T\langle B\rangle$, then $T_1+T_2 = T\langle A \cup B\rangle$.

Proof: Since $0 \in T_1$ and $0 \in T_2$, $T_1 \cup T_2 \subset T_1+T_2$. T_1+T_2 is an ideal. Let σ be any F-endomorphism of $F[X]$, and $t_1+t_2 \in T_1+T_2$ where $t_1 \in T_1$ and $t_2 \in T_2$. Then $(t_1+t_2)^\sigma = t_1^\sigma+t_2^\sigma \in T_1+T_2$, since T_1 and T_2 are T-ideals. Thus T_1+T_2 is a T-ideal and $T\langle T_1 \cup T_2\rangle \subset T_1+T_2$.

Since ideals are closed under addition, $T_1+T_2 \subset T\langle T_1 \cup T_2\rangle$ and thus $T_1+T_2 = T\langle T_1 \cup T_2\rangle$.

$A \cup B \subset T_1 \cup T_2 \subset T_1+T_2$. Let $t_1+t_2 \in T_1+T_2$, where $t_1 \in T_1$ and $t_2 \in T_2$. Then t_1 is based on A and t_2 is based on B so t_1+t_2 is based on $A \cup B$. But t_1+t_2 was arbitrary so $T_1+T_2 \subset T\langle A \cup B\rangle$ and by Lemma 2, $T_1+T_2 = T\langle A \cup B\rangle$.

A T-ideal T is <u>finitely-based</u> if there exists a finite set A which generates T.

Corollary 16: If T_1 and T_2 are finitely based T-ideals in $F[X]$, then T_1+T_2 is finitely based.

CHAPTER II

PRODUCT VARIETIES

We have spoken of intersection and union of varieties. We can also take a product of varieties. Let $\mathcal{M}$ and $\mathcal{N}$ be two varieties of F-algebras. Then the product $\underline{\mathcal{M}*\mathcal{N}}$ is the class of F-algebras which are the extensions of an algebra in $\mathcal{M}$ by an algebra in $\mathcal{N}$. The algebra $R \in \mathcal{M}*\mathcal{N}$ means there exists an ideal M in R so that $M \in \mathcal{M}$ and $R/M \in \mathcal{N}$. Since the trivial algebra is in any variety, both $\mathcal{M}$ and $\mathcal{N}$ are contained in $\mathcal{M}*\mathcal{N}$.

Lemma 17: $\mathcal{M}*\mathcal{N}$ is a variety.

Proof: We must show $\mathcal{M}*\mathcal{N}$ is closed. Let $R \in Q(\mathcal{M}*\mathcal{N})$; then there is an $R' \in \mathcal{M}*\mathcal{N}$ so that $R = R'/D$ for some ideal D in R'. But if $R' \in \mathcal{M}*\mathcal{N}$, then there exists an ideal M of R' so that $M \in \mathcal{M}$ and $R'/M \in \mathcal{N}$. Both M and D ideals means $M+D$ is an ideal. $M+D$ contains M so $R'/(M+D) \approx (R'/M)/(M+D/M)$. Thus $R'/(M+D)$ is a homomorphic image of R'/M, so $R'/(M+D)$ is also in $\mathcal{N}$. Further, $(D+M)/D \approx M/(D \cap M)$, so $(D+M)/D$ is a homomorphic image of M and thus in $\mathcal{M}$. But then $R = R'/D \in \mathcal{M}*\mathcal{N}$. Thus $Q(\mathcal{M}*\mathcal{N}) \subset \mathcal{M}*\mathcal{N}$ and $\mathcal{M}*\mathcal{N}$ is Q-closed.

Let $R \in S(\mathcal{M}*\mathcal{N})$. Then there is an $R' \in \mathcal{M}*\mathcal{N}$ so that $R < R'$. There exists an ideal M of R' so that $R'/M \in \mathcal{N}$ and $M \in \mathcal{M}$. $M \cap R$ is an ideal of R, and $M \cap R$ is a subalgebra of M so it is in $\mathcal{M}$. Further, $R/(M \cap R) \approx (M+R)/M$ a subalgebra of R'/M, so it is in $\mathcal{N}$. Thus $R \in \mathcal{M}*\mathcal{N}$ and $\mathcal{M}*\mathcal{N}$ is S-closed.

Let $R \in \Pi(\mathcal{M}*\mathcal{N})$. Then $R = \prod_\lambda R_\lambda$ where $R_\lambda \in \mathcal{M}*\mathcal{N}$ for all λ. Then for all λ, there is an ideal M_λ of R_λ so that $R_\lambda/M_\lambda \in \mathcal{N}$ and $M_\lambda \in \mathcal{M}$. Then $\prod_\lambda R_\lambda/M_\lambda \in \mathcal{N}$ and $\prod_\lambda M_\lambda \in \mathcal{N}$. But $\prod_\lambda (R_\lambda/M_\lambda) \approx \prod_\lambda R_\lambda/\prod_\lambda M_\lambda$, and thus $M = \prod_\lambda M_\lambda$ is an ideal of R and $R/M \in \mathcal{N}$ and $M \in \mathcal{M}$. So $R \in \mathcal{M}*\mathcal{N}$ and $\mathcal{M}*\mathcal{N}$ is Π-closed, so it is a variety.

Let T_1 and T_2 be T-ideals of $F[X]$. Then $\underline{T_1(T_2)} = T < \{t(f_1,\ldots,f_n) : t = t(s_1,\ldots,s_n) \in T_1 \text{ and } f_i \in T_2\}>$.

Lemma 18: The T-ideal of the variety $\mathcal{M}*\mathcal{N}$ is $T_{\mathcal{M}}(T_{\mathcal{N}})$.

Proof: Let $R \in \mathcal{M}*\mathcal{N}$. We shall show that R satisfies $T_{\mathcal{M}}(T_{\mathcal{N}})$. Let $g \in$ basis of $T_{\mathcal{M}}(T_{\mathcal{N}}) : g = t(f_1,\ldots,f_n)$ where $t \in T_{\mathcal{M}}$ and $f_i \in T_{\mathcal{N}}$, $i = 1,\ldots,n$. $g(R) = t(f_1(R),\ldots,f_n(R))$. Also there is an ideal M of R so that $R/M \in \mathcal{N}$ and $M \in \mathcal{M}$. f_i is an identity of R/M, so $f_i(R) \subset M$. Thus $g(R) \subset t(M)$. But t is an identity of M, so $t(M) = 0$. Thus g is an identity of R, but since g was an arbitrary element of a basis of $T_{\mathcal{M}}(T_{\mathcal{N}})$, R must satisfy all the identities of $T_{\mathcal{M}}(T_{\mathcal{N}})$. So $T_{\mathcal{M}}(T_{\mathcal{N}}) \subset T_{\mathcal{M}*\mathcal{N}}$.

Now let R satisfy the identities of $T_{\mathcal{M}}(T_{\mathcal{N}})$. Let $M = \bigcup_{f \in T_{\mathcal{N}}} f(R)$ and let μ and $\nu \in M$; $\mu = f(r_1,\ldots,r_m)$ and $\nu = g(r_{m+1},\ldots,r_{m+n})$ where f and $g \in T_{\mathcal{N}}$. Since $T_{\mathcal{N}}$ is an ideal, $f+g = (f+g)(x_1,\ldots,x_{m+n}) \in T_{\mathcal{N}}$ and thus $\mu+\nu \in M$. Similarly for $-\mu$ and $r\mu$ where $r \in R$. Thus M is an ideal. Obviously $R/M \in \mathcal{N}$ and since R satisfies $T_{\mathcal{M}}(T_{\mathcal{N}})$, M must satisfy $T_{\mathcal{M}}$ and thus $M \in \mathcal{M}$. Thus $R \in \mathcal{M}*\mathcal{N}$ and $T_{\mathcal{M}*\mathcal{N}} \subset T_{\mathcal{M}}(T_{\mathcal{N}})$ and so are equal.

Unfortunately Lemma 18 does not give us a method of finding

the identities of an F-algebra. Let R be an F-algebra and M an ideal of R. Suppose we know that the identities of M and R/M are T_1 and T_2 respectively and further that $M = \bigcup_{f \in T_2} f(R)$. Then by Lemma 18 we know that R is in the variety $\mathcal{V}_1 * \mathcal{V}_2$ corresponding to $T_1(T_2)$. In general, however, $T_1(T_2) \neq I(R)$. Consider the following ring of matrices:

$$R = \left\{ \begin{pmatrix} 0 & a_1 & a_2 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix} : a_1, a_2, a_3, a_4 \in F \text{ of char } 0 \right\}.$$

Let M be the ideal with $a_1 = a_3 = a_4 = 0$.

$$I(R/M) = T\langle xy-yx\rangle = T_2$$

$$I(M) = T\langle xy\rangle = T_1$$

By straightforward computation we find $xyz - zxy$ to be an identity of R. Also

$$R' = \left\{ \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} : a_1, a_2, a_3 \in R \right\}$$

with the ideal $N = \left\{ \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix} \right\}$ is in the variety $\mathcal{V}_1 * \mathcal{V}_2$. R' does not satisfy $xyz - zxy$ and thus $xyz - zxy \notin T_1(T_2)$. Thus $T_1(T_2) \neq I(R)$, although $I(R) \supset T_1(T_2)$.

The product of varieties unfortunately is non-associative as the following shows.

Lemma 19: $M_1*(M_2*M_3) \subseteq (M_1*M_2)*M_3$, where M_1, M_2 and M_3 are varieties of F-algebras.

Proof: Let $R \in M_1*(M_2*M_3)$. Then there exists an ideal M of R such that $R/M \in M_2*M_3$ and $M \in M_1$. Since $R/M \in M_2*M_3$, there exists an ideal N of R/M such that $(R/M)/N \in M_3$ and $N \in M_2$. But N an ideal of R/M means there exists an ideal K of R so that $N = K/M$. Now $K/M = N \in M_2$ and $M \in M_1$ means $K \in M_1*M_2$. Further $K \lhd R$ and $R/K \cong (R/M)/(K/M) \in M_3$ means $R \in (M_1*M_2)*M_3$, which proves the lemma.

This lemma can be restated in terms of T-ideals:

Corollary 19: $T_1(T_2(T_3)) \supseteq [T_1(T_2)]T_3$ for T_1, T_2, T_3 arbitrary T-ideals of $F[X]$.

Theorem 20: (Gavrilov [6]) The multiplication of varieties of associative F-algebras is non-associative.

Proof: Let M be the variety of commutative F-algebras and N the variety of F-algebras with null product, where F is of characteristic 0. Using commutator notation, i.e., $[a,b] = ab-ba$, we have $T_M = T\langle[x_1,x_2]\rangle$ and $T_N = T\langle x_1x_2\rangle$.

Obviously $T_M(T_N) \supseteq T\langle[x_1x_2,x_3x_4]\rangle$. Let $T_1 = T_M(T_M(T_N))$, then the identity $\phi(x_1,x_2,\ldots,x_8,y_1,y_2) = [[x_1x_2,x_3x_4]y_1,y_2[x_5x_6,x_7x_8]]$ is in T_1.

We shall show there exists a $G \in (M*M)*N$ which does not satisfy the identity ϕ and thus $G \notin M*(M*N)$.

Let G be the Grassman algebra of alternating multilinear forms on a free F-module of rank 14. An alternating form of degree

n is a function from $V \times V \times \ldots \times V$ to F such that $f(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n) = -f(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n)$. Then $G = \Lambda^1(V) \oplus \ldots \oplus \Lambda^{14}(V)$, the module direct sum, where $\Lambda^i(V)$ is acting on V^i and is also an F-module.

Multiplication in a Grassman algebra has two important properties:

Letting $a_1 \in \Lambda^r(V)$ and $a_2 \in \Lambda^s(V)$ then

(1) $a_1a_2 = (-1)^{rs}a_2a_1$

(2) $a_1a_2 \in \Lambda^{r+s}(V)$ and thus $a_1a_2 = 0$ if $r+s > 14$.

Let $M = \Lambda^2(V) \oplus \Lambda^3(V) \oplus \ldots \oplus \Lambda^{14}(V)$ and $N = \Lambda^6(V) \oplus \Lambda^8(V) \oplus \Lambda^9(V) \oplus \Lambda^{10}(V) \oplus \ldots \oplus \Lambda^{14}(V)$. By property (2) we see that N is an ideal of M which is an ideal of G. Again by property (2) G/M is null.

Using both properties we can show that both N and M/N are members of $\mathcal{M}$. Since G is a direct sum of F-modules we need only show it for elements of the summands. Let $a_1 \in \Lambda^r(V) \cap M$ and $a_2 \in \Lambda^s(V) \cap M$. If r or s is even, then $[a_1,a_2] = 0$. If both r and s are odd, then $r+s$ is even. Both r and s are greater or equal to 3 and so $r+s$ is even and greater or equal to 6. Thus $[a_1,a_2] \in N$ and M/N is commutative. Similarly we can show N is commutative. Thus $M \in (\mathcal{M}*\mathcal{M})$ and $G \in (\mathcal{M}*\mathcal{M})*\mathcal{N}$.

On the other hand, by making the following substitution in ϕ we shall show $\phi(a_i) \neq 0$.

Let $0 \neq a_i \in G$ and $a_1,a_3,a_5,a_7 \in \Lambda^2(V)$, $a_i \in \Lambda^1(V)$

otherwise. We have $\phi(a_1,\ldots,a_{10}) = [[a_1a_2,a_3a_4]a_9,a_{10}[a_5a_6,a_7a_8]]$. Now a_1a_2, a_3a_4, a_5a_6 and a_7a_8 are elements of $\Lambda^3(V)$. Thus $[a_1a_2,a_3a_4] = 2a_1a_2a_3a_4 \neq 0$ and in $\Lambda^6(V)$. Also $[a_1a_2,a_3a_4]a_9 = 2a_1a_2a_3a_4a_9 \in \Lambda^7(V)$. Similarly $a_{10}[a_5a_6,a_7a_8] \in \Lambda^7(V)$ and is not zero. So $\phi(a_i) = 8a_1a_2a_3a_4a_9a_{10}a_5a_6a_7a_8 \neq 0$ and in $\Lambda^{14}(V) \subset G$. Thus G is not in $\mathcal{M}*(\mathcal{M}*\mathcal{N})$ and multiplication of varieties is not associative.

CHAPTER III

IDENTITIES AND FINITE BASIS THEOREMS

Now we shall examine the identities themselves. Any polynomial $f \in F[X]$ is a unique linear combination of monomials in the noncommuting indeterminates in X. The monomials with nonzero coefficients in this expression are called the <u>monomials of f</u>. A monomial is of <u>degree</u> d_i <u>in</u> x_i if x_i occurs d_i times in that monomial; f is of degree d_i in x_i if $d_i = \max$ {degree in x_i of each monomial of f}.

The polynomial f is <u>homogeneous</u> in x_i if all monomials of f have the same degree in x_i; f is completely homogeneous if f is homogeneous in every x_i. Call f <u>blended</u> in x_i if x_i appears in every monomial of f; f is blended if it is blended in each x_i which occurs in f. The polynomial f is <u>linear</u> in x_i if each monomial is of degree one in x_i; f is multilinear if it is linear in every x_i which occurs in f.

Lemma 21: Let R be an F-algebra with a polynomial identity f. Then $f = \sum f_\alpha$, where each f_α is blended and an identity for R.

Proof: Let x be a variable occuring in f. $f(x) = f_1+f_2$, where f_1 is blended in x and f_2 is of degree 0 in x. Then $f(0) = 0+f_2$ and so $f_2 \in I(R)$. But $f-f_2 = f_1 \in I(R)$ also. Repeat process on each new identity and for each variable until all new identities are blended.

Thus each polynomial identity is based on blended identities; and in terms of T-ideals we have:

Corollary 21: A T-ideal is based on its blended polynomials.

The <u>degree of f</u> is max $\{\sum_i$ degree in x_i of each monomial of $f\}$.

Theorem 22: (Kaplansky [11]) Any ring satisfying a polynomial identity f of degree d, satisfies a multilinear identity g of degree $\leq d$, with coefficients from the set of coefficients in f.

Proof: The ring R is an F-algebra for some ring F. By Lemma 21 we can assume f is blended. Assume that the degree of $f(x_1,x_2,\ldots,x_n)$ in x_1 is greater than one. Define

$$f_{1,2} = f(y_1+y_2,x_2,\ldots,x_n) \quad \text{substitute } y_1+y_2 \text{ for } x_1$$

$$f_1 = f(y_1,x_2,\ldots,x_n) \quad \text{substitute } y_1 \text{ for } x_1$$

$$f_2 = f(y_2,x_2,\ldots,x_n) \quad \text{substitute } y_2 \text{ for } x_1 .$$

Let $g = f_{1,2}-f_1-f_2 = g(y_1,y_2,x_2,\ldots,x_n)$. Consider any monomial m of f such that degree, d, of m in x_1 is greater than one. $m = cm_o x_1 m_1 x_1 \ldots x_1 m_d$ where $m_i = 1$ or a product of variables all distinct from x_1; $c \in F$. The substitution of y_1+y_2 for x_1 gives us

$$cm_o(y_1+y_2)m_1(y_1+y_2)\ldots(y_1+y_2)m_d$$

$$= \sum_{\substack{i(j) = 1 \text{ or } 2 \\ \text{distinct } (i(1),i(2),\ldots,i(d))}} cm_o y_{i(1)} m_1 y_{i(2)} \cdots y_{i(d)} m_d \ .$$

Thus $f_{1,2}$ has monomials in which both y_1 and y_2 occur and so must g. Thus g is nontrivial. Further all monomials in $f_{1,2}$ in which y_i does not occur are exactly those of f_j, $i \neq j$, $i = 1$ or 2, $j = 1$ or 2. Thus g is blended in y_1 and y_2. For $i \neq 1$, f is blended in x_i so g is blended in x_i. Thus g is blended in all variables and the coefficients of g are among those of f. The degree of $f_{1,2}$ and g is not increased by this substitution, so degree $g \leq d$. g, however, has one more variable than f.

If g is not multilinear, repeat process on a variable of degree greater than one until the number of variables is equal to the degree of the derived polynomial. This must be a multilinear identity.

If at each step a variable is chosen which occurs at least twice in a monomial of degree d, then a multilinear identity of degree d is obtained.

If the process of Theorem 22 is applied to a variable, say x, of a polynomial f, and to all variables replacing x until they are all linear we shall say x is <u>linearized</u>.

Theorem 23: Let F be a field and $f(x_1,\ldots,x_n)$ be a polynomial in $F[X]$ which is homogeneous in some variable, say x_1 of degree d. Let g be the polynomial derived from f by the linearization of x_1. If x_1 is subtituted for the new variables in g, then $g(x_1,x_1,\ldots,x_1,x_2,\ldots,x_n) = d!f(x_1,x_2,\ldots,x_n)$. Further if the characteristic of F is zero or greater than d, then f is based on g.

Proof: Inductively define

$$f_{i+1} = f_i(x_1+y_{i+1},y_1,\ldots,y_i,x_2,\ldots,x_n)$$
$$- f_i(x_1,y_1,\ldots,y_i,x_2,\ldots,x_n)$$
$$- f_i(y_{i+1},y_1,\ldots,y_i,x_2,\ldots,x_n)$$

as in Theorem 22 for $i \leq d-2$, $f = f_o$.

Let $f_i = g_i+h_i$, where g_i contains all the monomials of degree $d-i$ in x_1 and is linear in $y_1,\ldots,y_i$. Since f is homogeneous in x_1, $f = f_o = g_o$. Note that f_i, g_i and h_i are functions of $x_1,y_1,\ldots,y_i,x_2,\ldots,x_n$. Also f_i is based on f_o for all i.

I shall show that $g_{i+1}(x_1,y_1,\ldots,y_i,x_1,x_2,\ldots,x_n) = (d-i)g_i(x_1,y_1,\ldots,y_i,x_2,\ldots,x_n)$ and thus $g = g_{d-1}(x_1,\ldots,x_1,x_2,\ldots,x_n) = d!g_o(x_1,\ldots,x_n)$.

Let m_{i+1} be any monomial of g_{i+1}. Thus m_{i+1} has degree $d-(i+1)$ in x_1. Let it come from monomial m_i of f_i. Suppose m_i is of degree $d-(i+1)$ in x_1. Then $m_i = m_{i+1}$ and thus was eliminated from f_{i+1}. Thus m_i is of degree $d-i$ in x_1 and is a monomial of g_i. If this is so, g_{i+1} must be linear in y_{i+1}. Further each monomial of g_i is replaced by one monomial for each place y_i can occur. Thus each monomial of g_i is replaced by $d-i$ monomials in g_{i+1}. Replacing y_{i+1} by x_1 gives $g_{i+1}(x_1,y_1,\ldots,y_i,x_1,x_2,\ldots,x_n) = (d-i)g_i(x_1,y_1,\ldots,y_i,x_2,\ldots,x_n)$.

The degree of y_j, $j \leq i$, does not differ in g_i and g_{i+1}, nor does the degree of x_j, $j \neq 1$. Thus $g(x_1,\ldots,x_1,x_2,\ldots,x_n) = d!f(x_1,x_2,\ldots,x_n)$.

If the characteristic of F is 0 or greater than d, then $d!$ is invertible in F and f is based on g.

Suppose f is not homogeneous. Then g_o is based on g_{d-1} which is based on f. So $f-g_o = h_o$ is also based on f, and the process of Theorem 23 can be applied to h_o and x_1, and reapplied as many times as necessary. Let $f = \sum_i f_i$, where each f_i is homogeneous of degree i in x_1. Then each f_i is based on g_i in which x_1 is linearized, and each g_i is based on f.

Theorem 23 can then be applied to the other variables to give the following corollary.

Corollary 23: Let $f(x_1,\dots,x_n)$ be any polynomial such that the characteristic of the field F is zero or greater than $\max\{d_i\}$. Then $T\langle f\rangle$ is based on multilinear identities, each based on f.

Theorem 24: Let f be a polynomial of degree d; $n = [\frac{d}{2}]+1$. Let R be a ring such that no coefficient of f annihilates R^d; then $(R)_n$, the $n \times n$ matrix ring over R, does not satisfy f.

Proof: By Theorem 18, if f is an identity of $(R)_n$ then there is a multilinear identity g of degree d or less with coefficients from those of f. Then g can be written as $c_1x_1\dots x_d + \sum_{id\neq\sigma} c_\sigma x_{\sigma(1)}\dots x_{\sigma(d)}$, where σ is a permutation of the d variables and $c_1R^d \neq 0$.

There exist $r_1,\dots,r_d \in R$ so that $c_1r_1\dots r_d \neq 0$. Substitute the matrices $r_1e_{1,1}, r_2e_{1,2}, r_3e_{2,2},\dots$ for $x_1,\dots,x_d$ respectively, ending with $r_de_{n-1,n}$ if d is even and $r_de_{n,n}$ if d is odd; where $e_{i,j}$ are matrix units.

Then $c_1x_1 \ldots x_d = c_1r_1 \ldots r_d e_{1,n} \neq 0$ but for any other permutation of these matrix units their product will be 0.

Thus $g = c_1r_1, \ldots, r_d e_{1,n} \neq 0$ and f cannot be an identity for $(R)_n$.

The hypothesis of this theorem could be weakened as long as a multilinear identity is obtained so that c is a coefficient of some monomial in that multilinear identity and $cR^d \neq 0$. The hypothesis could be: Let R be a ring such that for some coefficient c of f so that the degree of some variable of that monomial is maximum, c does not annihilate R^d. But even this hypothesis is too strong.

There is a very important corollary to this theorem.

Corollary 24: There does not exist a universal identity for all matrix algebras over an algebra with unity.

Theorem 25: (Kaplansky [11]) Let R be an algebra over a field F with at least $n+1$ elements, satisfying an identity of degree at most n in each of its m indeterminates. Then $f = \sum f_\alpha$, where each f_α is completely homogeneous and R satisfies each f_α.

Proof: Assume f is blended. If f is not completely homogeneous assume it is not homogeneous in x_i; degree of $x_i \leq n$. Then $f = \sum_{j=1}^{n} f_j$ where f_j is of degree j in x_i. Make n distinct substitutions of $c_k x_i$ for x_i where $0 \neq c_k \in F$. Then

$$f(x_1, \ldots, c_k x_i, \ldots, x_m) = \sum_{j=1}^{n} c_k^j f_j(x_1, \ldots, x_i, \ldots, x_m) \quad .$$

Construct the determinant of coefficients c_k^j. This is a Vandermonde determinant in $c_1,\ldots,c_n$ and thus not zero. This system of equations is solvable and each f_j is an identity for R. Repeat for each nonhomogeneous variable.

Lemma 26: (Kaplansky [11]) If an algebra A satisfies a polynomial identity, then it satisfies a polynomial identity in two variables.

Proof: Let $f = f(x_1,\ldots,x_n)$ be a polynomial identity having more than one variable. For x_i substitute $y^i z$ where y and z are noncommuting variables.

$$g(y,z) = f(yz, y^2 z, \ldots, y^n z) \quad .$$

If f has only one variable, substitute $y+z$ for that variable.

The next few theorems pertain to F-algebras where F is a field of characteristic 0.

Theorem 27: Any T-ideal in $F[X]$, where F is a field of characteristic 0, is based on its multilinear identities.

Proof: By Corollary 23, each polynomial is based on multilinear identities. Thus any T-ideal is based on multilinear identities.

Since each identity is based on a finite number of completely

homogeneous identities, and each completely homogeneous identity on one multilinear identity, we have the following important corollary:

Corollary 27: If a T-ideal of $F[X]$, where F is of characteristic 0, is finitely based, then it is finitely based on multilinear identities.

Theorem 28: Let F be a field of characteristic 0, T_1 and T_2 be finitely based T-ideals in $F[X]$. Then there exists a finite basis for $T_1(T_2)$.

Proof: By Corollary 27, T_1 has a finite basis of multilinear identities, call it A. Let B be a finite basis for T_2.

Let $B' = \{x_1^{\epsilon_1} g x_2^{\epsilon_2} : \epsilon_i = 0$ or 1, $g \in B$ and x_1 and x_2 are variables not appearing in $g\}$.

Let $C = \{f(g_1', \ldots, g_n') : f \in A,\ g_i' \in B'\}$.

We shall show that $T_1(T_2) = T\langle C\rangle$. Let

$$D = \{f(t_1, \ldots, t_n) : f(x_1, \ldots, x_n) \in A,\ t_i \in T_2\} .$$

Let

$$E = \{h(t_1, \ldots, t_n) : h(x_1, \ldots, x_n) \in T_1,\ t_i \in T_2\} .$$

By definition $T\langle E\rangle = T_1(T_2)$.

We shall show $E \subset T\langle D\rangle$ and $D \subset T\langle C\rangle$ and thus by Lemma 2, $T\langle E\rangle = T\langle C\rangle$.

Let $e \in E$ such that $e = h(t_1, \ldots, t_n)$ for $h \in T_1$, $t_i \in T_2$. But since $h \in T_1$, $h = \sum_i k_i \alpha_i^{\epsilon_{i,1}} f_i(\gamma_{i,j}) \beta_i^{\epsilon_{i,2}}$ where

$\alpha_i, \beta_i, \gamma_{i,j} \in F[X]$, $f_i \in A$ and $k_i \in F$

and $\epsilon_{i,1}$ and $\epsilon_{i,2}$ are 0 or 1 .

By definition of a T-ideal, $\gamma_{i,j}(t_\ell) = t_{i,j} \in T_2$ if $\{t_\ell\} \in T_2$. Certainly $\beta_i(t_\ell) = \beta_i' \in F[X]$ and $\alpha_i(t_\ell) = \alpha_i' \in F[X]$.

Thus $e = h(t_1,\ldots,t_n) = \sum_i k_i \alpha_i'^{\epsilon_{i,1}} f_i(t_{i,1},\ldots,t_{i,n(i)}) \beta_i'^{\epsilon_{i,2}}$ is in T<D>, since $f_i(t_{i,1},\ldots,t_{i,n(i)}) \in D$. $D \subset E$ obviously and thus T<E> = T<D>.

Let $d \in D$, $d = f(t_1,\ldots,t_n)$, where $f \in A$, $t_i \in T_2$. Let $t_1 = \sum_i k_i \alpha_i^{\epsilon_{i,2}} b_i \beta_i^{\epsilon_{i,2}}$, where $b_i \in B$,

$\alpha_i, \beta_i \in F[X]$, $k_i \in F$ and $\epsilon_{i,j} = 0$ or 1 .

Thus $f(t_1,\ldots,t_2) = f(\sum_i \alpha_i^{\epsilon_{i,1}} b_i \beta_i^{\epsilon_{i,2}}, t_2,\ldots,t_n)$. Since f is multilinear, the summation sign can be brought out. So

$$d = f(t_1,\ldots,t_n) = \sum_i f(\alpha_i^{\epsilon_{i,1}} b_i \beta_i^{\epsilon_{i,2}}, t_2,\ldots,t_n) \quad .$$

Repeat process on $t_2,\ldots,t_n$ for each polynomial within the summation sign.

Remembering that each variable in a polynomial in the generator can be replaced by any polynomial in $F[X]$ and then the polynomials can be summed, we see that $d \in$ T<C>. Thus $D \subset$ T<C> and since $C \subset D$, it must be that T<D> = T<C>. So we have $T_1(T_2)$ = T<E> = T<D> = T<C>.

Further, we see that given bases for T_1 and T_2 in $F[X]$, where F is of characteristic 0, a basis for $T_1(T_2)$ can be computed. For example, in Theorem 20, $T_M(T_N) = T\langle[x_1x_2,x_3x_4]\rangle$, since the extra variables can be absorbed by x_1, x_2, x_3 or x_4.

Also

$$T_1 = T <\{[y_1^{\epsilon_1}[x_1x_2,x_3x_4]y_2^{\epsilon_2}, y_3^{\epsilon_3}[x_5x_6,x_7x_8]y_4^{\epsilon_4}]$$

$$\text{where} \quad \epsilon_i = 0 \quad \text{or} \quad 1\}>$$

We have other criteria for a finite basis.

Theorem 29: (D.E. Cohen [4]) Any T-ideal containing $[x,y]$ is finitely generated.

The standard identity on n variables denoted as S_n or $[x_1 \ldots x_n]$ is the multilinear identity:

$$\sum_{\sigma \epsilon P_n} (-1)^{\sigma} x_{\sigma(1)} \cdots x_{\sigma(n)}$$

where P_n is the permutation group on n objects and $(-1)^{\sigma}$ is equal to -1 or 1 if σ is an odd or even permutation respectively. In some sense, the standard identity is a generalization of the commutative identity.

We have the further criterion.

Theorem 30: (V.N. Latychev [12]) Any T-ideal containing S_4 is finitely generated.

CHAPTER IV

SOME SPECIFIC RESULTS

We have a few specific results which may help us in computing identities of a ring.

Lemma 31: An F-algebra, R, where F is a field of characteristic 0, which satisfies a polynomial identity is either nil or satisfies a polynomial in $T\langle xy-yx \rangle$.

Proof: By Theorem 22, R satisfies a multilinear identity of degree d : say $f(x_1,\ldots,x_d) = \sum_\sigma c_\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(d)}$, where c_σ are in F and not all are zero, and σ varies over the permutations of d objects.

Set $x_1 = x_2 = \ldots = x_d = x$. Then

$$f(x,x,\ldots,x) = \sum_\sigma c_\sigma x^d .$$

If $\sum_\sigma c_\sigma$ is not zero, then it is invertible in F and $x^d = 0$. Thus R is nil of exponent d. If $\sum_\sigma c_\sigma = 0$, then let us look at $\sum_\sigma c_\sigma x_{\sigma(1)} \cdots x_{\sigma(d)} + T$ where $T = T\langle xy-yx \rangle$. We know that $F[X]/T\langle xy-yx \rangle$ is commutative. So $\sum_\sigma c_\sigma x_{\sigma(1)} \cdots x_{\sigma(d)} + T = \sum_\sigma c_\sigma x_1 \cdots x_d + T = 0 + T$ and thus is in T.

Theorem 32: An F-algebra of finite dimension, n, over F satisfies the standard identity on n+1 variables.

Proof: We need only prove it for the n elements of a basis. A standard identity is zero if any two variables are the

same. There are, however, only n elements of the basis to substitute for the $n+1$ variables, and thus two of them must be the same.

My original problem was to find the identities of a certain ring. This is very difficult and in general impossible. Amitsur and Levitzki have worked on complete matrix rings with partial results.

Theorem 33: (Amitsur and Levitzki [3]) An $n \times n$ matrix ring over a commutative ring, satisfies the standard identity on $2n$ variables.

For a nice proof see Swan [13].

Let us take the simple case of F being a field of characteristic 0 and $F_{n\times n}$ the $n \times n$ matrix ring over F.

Theorem 34: (Amitsur and Levitzki [3]) If $F_{n\times n}$ satisfies a multilinear identity f of degree $2n$, then $f = c\,S_{2n}$, where $c \in F$.

Further, we know by Theorem 24 that $F_{n\times n}$ does not satisfy a polynomial identity of degree less than $2n$. It has been known for some time that $[[x,y]^2,z]$ is a polynomial identity for 2×2 matrices (Herstein [8], p. 153). More recently, Formanek [5] has generalized this identity for $n \times n$ matrices. These are very complicated identities but they seem to show that the standard identity on $2n$ variables does not generate the T-ideal of identities of $F_{n\times n}$.

Lemma 35: Let p and q be any polynomials and x,y,z variables. Then $px[y,z]q = p[x,y]zq+p[y,xz]q$.

The proof is by direct computation.

Lemma 36: Let $T_1 = T\langle x_1 \dots x_n\rangle$ and $T_2 = T\langle [x,y]\rangle$, then $T_1(T_2) = T\langle [x_1,x_2][x_3,x_4] \dots [x_{2n-1},x_{2n}]\rangle = T_3$.

Proof: By Theorem 28, $T_1(T_2)$ is based on

$$C = \{[x_1,x_2]y_1^{\epsilon_1}[x_3,x_4]\dots y_{n-1}^{\epsilon_{n-1}}[x_{2n-1},x_{2n}] : \epsilon_i = 0 \text{ or } 1\}.$$

By repeated application of Lemma 35 we find that each element of C is in T_3.

We do have a very few tools for finding identities for F-algebras where F is of characteristic 0.

Lemma 37: Let f be an identity such that each monomial is of degree at least d, and g an identity of degree less than d. Then g is not based on f.

Proof: Certainly addition and subtraction does not decrease the degrees of the monomials. Also the image of a monomial under an endomorphism of $F[X]$ would be either zero or of greater or equal degree. So g is not based on f.

One can look at identities of subrings and see if these hold for the whole ring. If an ideal can be found and the identities of the ideal and the quotient ring found, then by Lemma 18 some identities for the ring can be found.

Consider my original problem: find the polynomial identities of the upper triangular 3×3 matrices over the rationals with equal entries along the diagonal.

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a,b,c,d \in Q = \text{rationals} \right\}$$

The ideal

$$C = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : c \in Q \right\}$$

is null and so belongs to the variety, V_1, associated with $T_1 = T\langle xy \rangle$. Further R/C belongs to the variety V_2 of commutative Q-algebras. Thus $R \in V_1 * V_2$ associated with $T_1(T_2)$ where $T_2 = T\langle xy-yx \rangle$. By Lemma 36, $T_1(T_2) = T\langle [x_1,x_2][x_3,x_4] \rangle$ and thus $p = [x_1,x_2][x_3,x_4]$ is an identity of R.

One can also note that C is in the center of R. Therefore $q = [[x_1,x_2],x_3]$ is also an identity for R.

By Lemma 37, the identity q is not based on p. In a Grassman algebra of alternating multilinear forms on a free Q-module of rank 4, $[[x_1,x_2],x_3] = 0$, since $[x_1,x_2]$ is either 0 or has an even degree. $[x_1,x_2][x_3,x_4]$ is not zero in general. Thus p is not based on q.

Let $m(x,y,z)$ be any multilinear identity of R. By substituting

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

for all permutations of x, y and z we find that $m \in T\langle[[x_1,x_2],x_3]\rangle$.

One notes that the standard identity, S_4 is based on $[x_1,x_2][x_3,x_4]$. Therefore by Theorem 30, $I(R)$ is finitely generated. Whether $T\langle[x_1,x_2][x_3,x_4],[[x,y]z]\rangle$ are all the identities of R, I do not know.

Let us consider now F-algebras containing unity. Let $F[X^1]$ be $F[X]$ extended by 1. The T-ideals of $F[X^1]$ are defined similarly, thus also allowing endomorphisms which take an indeterminate to 1. Corresponding to these T-ideals are the varieties of F-algebras containing unity. A class is S-closed if it contains all the F-subalgebras containing unity.

There are some differences. Lemma 37 fails. Indeed we have the following:

Lemma 38: In $F[X^1]$, $T\langle S_{2n+1}\rangle = T\langle S_{2n}\rangle$.

Proof: Consider the endomorphism taking $x_{2n+1} \to 1$, and identity elsewhere. Then

$$S_{2n+1}(x_1,\ldots,x_n,1) = \sum_{i=1}^{2n+1} c_i\ S_{2n}$$

where $c_i = 1$ or -1. $2n+1$ is odd and thus $\sum c_i \neq 0$. Thus $S_{2n} \in T\langle S_{2n+1}\rangle$. Obviously $S_{2n+1} \in T\langle S_{2n}\rangle$ and so they are equal.

Note that if unity is substituted for a variable in S_{2n}, then $\sum c_i = 0$, so $S_{2n-1} \notin T\langle S_{2n}\rangle$.

Corollary 38: If R is an F-algebra of dimension $2n$ over F and R contains unity, then R satisfies S_{2n}.

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